The Quantum Muon

$$\frac{1}{\sqrt{2}}\Bigg(\Bigg|$$  $$\Bigg\rangle_{\mu^+} + \Bigg|$$  $$\Bigg\rangle_{\mu^+}\Bigg)$$

Lecture 1: Spin and precession

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1

The muon is a quantum object.

These lectures therefore give an introduction to quantum-mechanical techniques used to describe muon experiments.

This lecture: spin precession.

First classical treatment, then several quantum methods.

2

State of the system described by $|\psi\rangle$

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}$$
ket

$$\langle\psi| = \begin{pmatrix} \psi_1^* & \psi_2^* & \dots & \psi_N^* \end{pmatrix}$$
bra

$$\langle\phi|\psi\rangle = \begin{pmatrix} \phi_1^* & \phi_2^* & \dots & \phi_N^* \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} = \phi_1^*\psi_1 + \dots \phi_N^*\psi_N = \text{complex number}$$
bra-c-ket

3

Magnetic moment and angular momentum

$$\vec{\mu} = \gamma \vec{L}$$

magnetic moment

angular momentum

gyromagnetic ratio $= \frac{gq}{2m}$

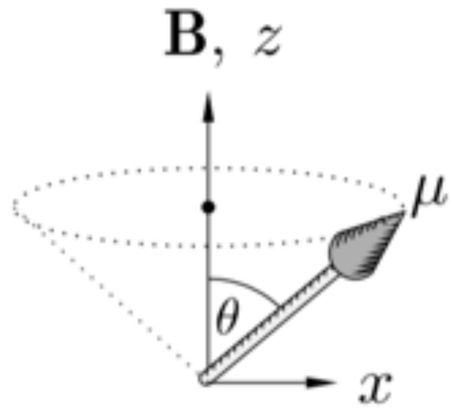
$$\vec{\mu}_\mu = \gamma_\mu \vec{S}_\mu \qquad \frac{\gamma_\mu}{2\pi} = 135.5 \text{ MHz T}^{-1}$$

4

Properties of particles

Property	μ^+	π^+	e	p
Mass	1.8835×10^{-28} kg	2.488×10^{-28} kg	9.1094×10^{-31} kg	1.6726×10^{-27} kg
	105.66 MeV	139.57 MeV	0.51100 MeV	938.27 MeV
	$0.1126 m_p$	$0.1487 m_p$	$m_p/1836.2$	m_p
	$206.768 m_e$	$273.13 m_e$	m_e	$1836.2 m_e$
Charge	$+e$	$+e$	$-e$	$+e$
Spin	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
Magnetic moment	$4.4904\times10^{-26}\ \mathrm{J\,T^{-1}}$	0	$-928.48\times10^{-26}\ \mathrm{J\,T^{-1}}$	$1.4106\times10^{-26}\ \mathrm{J\,T^{-}}$
	$3.1833 \mu_p$	0	$-658.21 \mu_p$	μ_p
	$8.891 \mu_N$	0	$-1838.3 \mu_N$	$2.7928 \mu_N$
	$4.842\times10^{-3} \mu_B$	0	$-1.001 \mu_B$	$1.521\times10^{-3} \mu_B$
$\lvert\gamma\rvert/(2\pi)$	$135.53\ \mathrm{MHz\,T^{-1}}$	0	$28024.21\ \mathrm{MHz\,T^{-1}}$	$42.577\ \mathrm{MHz\,T^{-1}}$
Lifetime	2.19703×10^{-6} s	0.0260×10^{-6} s	$> 4 \times 10^{23}$ years	$> 2 \times 10^{26}$ years

5



spin precession

E-fields don't do this to electric dipole moments

6

Classical treatment of spin precession

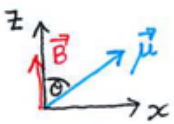
energy

$$E = -\vec{\mu}\cdot\vec{B}$$

$$\vec{G} = \vec{\mu}\times\vec{B}$$

torque

$$\frac{d\vec{\mu}}{dt} = \gamma\,\vec{\mu}\times\vec{B}$$



$$\begin{pmatrix}\dot{\mu}_x\\ \dot{\mu}_y\\ \dot{\mu}_z\end{pmatrix} = \begin{pmatrix}\gamma B\mu_y\\ -\gamma B\mu_x\\ 0\end{pmatrix} \Rightarrow \begin{aligned} \mu_x(t) &= |\vec{\mu}|\sin\theta\cos\omega t\\ \mu_y(t) &= -|\vec{\mu}|\sin\theta\sin\omega t\\ \mu_z(t) &= |\vec{\mu}|\cos\theta \end{aligned}$$

$\Rightarrow$ SPIN PRECESSION $\quad \omega = \gamma_\mu B$

7

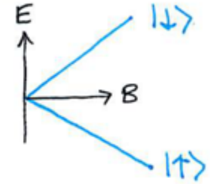
Quantum mechanics of spin-$\frac{1}{2}$

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle = \begin{pmatrix}a\\ b\end{pmatrix} \leftarrow \text{SPINOR}$$

$$|\uparrow\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$$\hat{H} = -\vec{\mu}\cdot\vec{B} = -\frac{\hbar}{2}\gamma B\sigma_z = -\frac{\hbar\omega}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$



$E = \frac{\hbar\omega}{2}$

$E = -\frac{\hbar\omega}{2}$

Zeeman splitting

8

Pauli spin matrices

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

For a general direction $\hat{\vec{n}}$

$$\hat{n}\cdot\vec{\sigma} = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

Eigenstates are $|+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}$ $|-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}$

9

Pauli spin matrices

σ_x	σ_y	σ_z
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Eigenstates

$\vert\uparrow_x\rangle$	$\vert\downarrow_x\rangle$	$\vert\uparrow_y\rangle$	$\vert\downarrow_y\rangle$	$\vert\uparrow_z\rangle$	$\vert\downarrow_z\rangle$
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

10

Quantum mechanical treatment of spin precession (1)

$B \parallel z$ Initial muon polarization

$$|\psi(0)\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle$$

Time-dependence : $\hat{H}|\psi\rangle = i\hbar\frac{d}{dt}|\psi\rangle$

$$|\psi(t)\rangle = e^{i\omega t/2}\cos\frac{\theta}{2}|\uparrow\rangle + e^{-i\omega t/2}\sin\frac{\theta}{2}|\downarrow\rangle$$

$$\left.\begin{aligned} \langle\psi(t)|\sigma_x|\psi(t)\rangle &= \sin\theta\cos\omega t \\ \langle\psi(t)|\sigma_y|\psi(t)\rangle &= -\sin\theta\sin\omega t \\ \langle\psi(t)|\sigma_z|\psi(t)\rangle &= \cos\theta \end{aligned}\right\}\ \text{spin precession}$$

11

Quantum mechanical treatment of spin precession (2)

Time-evolution operator $\hat{H} = -\frac{\hbar\omega}{2}\,\vec{n}\cdot\vec{\sigma}$

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = \cos\frac{\omega t}{2}\,I + i\sin\frac{\omega t}{2}\,\underbrace{\hat{n}\cdot\vec{\sigma}}_{\sigma_z}$$

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \begin{pmatrix} \cos\frac{\omega t}{2} + i\sin\frac{\omega t}{2} & 0 \\ 0 & \cos\frac{\omega t}{2} - i\sin\frac{\omega t}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\omega t/2}\cos\frac{\theta}{2} \\ e^{-i\omega t/2}\sin\frac{\theta}{2} \end{pmatrix} \quad \rightarrow \text{same result}$$

12

Density matrix

$$\rho = |\psi\rangle\langle\psi| \quad \text{.... pure state}$$

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle = \mathrm{Tr}\left(\hat{A}\,|\psi\rangle\langle\psi|\right) = \mathrm{Tr}(\hat{A}\rho)$$

example $\hat{A} = \text{identity}$, $\mathrm{Tr}\,\rho = 1$

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle \qquad \langle\psi(t)| = \hat{U}(t)^{\dagger}\langle\psi(0)|$$

$$\therefore\ \rho(t) = \hat{U}(t)\,\rho(0)\,\hat{U}(t)^{\dagger}$$

13

Pauli spin matrices

σ_x	σ_y	σ_z
$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Eigenstates

$\lvert\uparrow_x\rangle$	$\lvert\downarrow_x\rangle$	$\lvert\uparrow_y\rangle$	$\lvert\downarrow_y\rangle$	$\lvert\uparrow_z\rangle$	$\lvert\downarrow_z\rangle$
$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}$	$\begin{pmatrix}1\\ 0\end{pmatrix}$	$\begin{pmatrix}0\\ 1\end{pmatrix}$

Density matrices

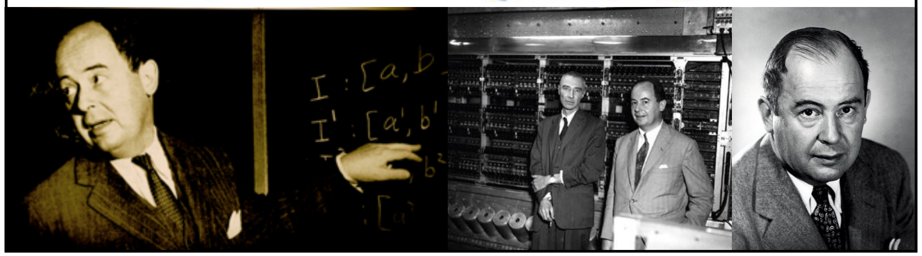
$\lvert\uparrow_x\rangle\langle\uparrow_x\rvert$	$\lvert\downarrow_x\rangle\langle\downarrow_x\rvert$	$\lvert\uparrow_y\rangle\langle\uparrow_y\rvert$	$\lvert\downarrow_y\rangle\langle\downarrow_y\rvert$	$\lvert\uparrow_z\rangle\langle\uparrow_z\rvert$	$\lvert\downarrow_z\rangle\langle\downarrow_z\rvert$
$\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

14

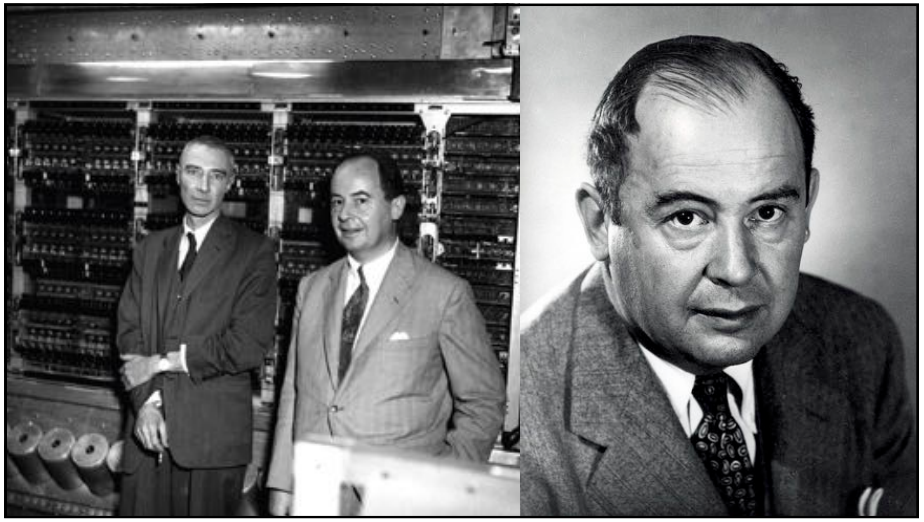
Density matrix for a spin-$\frac{1}{2}$ particle

$$\rho = \frac{1}{2}\begin{pmatrix}1+n_z & n_x - i n_y\\ n_x + i n_y & 1-n_z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+\cos\theta & \sin\theta\, e^{-i\phi}\\ \sin\theta\, e^{i\phi} & 1-\cos\theta\end{pmatrix}$$

$$= \frac{1}{2}\left(I + \vec{\sigma}\cdot\hat{n}\right)$$ (VERY USEFUL)



15



16

Quantum mechanical treatment of spin precession (3)

The initial muon spin is $\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$, so the density matrix is

$$\rho(0) = |\psi(0)\rangle\langle\psi(0)| = \begin{pmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix}$$

so that adding the time dependence produces

$$\rho(t) = \hat{U}(t)\rho(0)\hat{U}(t)^\dagger. = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{i\omega t}\sin\theta \\ e^{-i\omega t}\sin\theta & 1-\cos\theta \end{pmatrix}.$$

Then the expected value of the spin is given by

$$\langle\sigma_x(t)\rangle = \mathrm{Tr}(\sigma_x\rho(t)) = \sin\theta\cos\omega t$$
$$\langle\sigma_y(t)\rangle = \mathrm{Tr}(\sigma_y\rho(t)) = -\sin\theta\sin\omega t$$
$$\langle\sigma_z(t)\rangle = \mathrm{Tr}(\sigma_z\rho(t)) = \cos\theta.$$

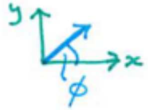
17

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$$

spin // x

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle\right)$$

rotated



$$\rho_{unpol} = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In general : $\boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$ mixed state

$\because \sum p_i = 1$, $\mathrm{Tr}\,\rho = 1$ still.

$$\langle\sigma_x\rangle = \mathrm{Tr}(\sigma_x\rho_{\mathrm{unpol}}) = 0$$
$$\langle\sigma_y\rangle = \mathrm{Tr}(\sigma_y\rho_{\mathrm{unpol}}) = 0$$
$$\langle\sigma_z\rangle = \mathrm{Tr}(\sigma_z\rho_{\mathrm{unpol}}) = 0.$$

18