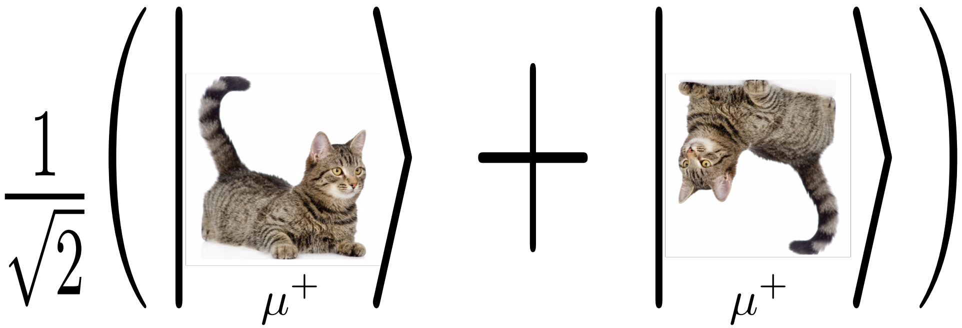
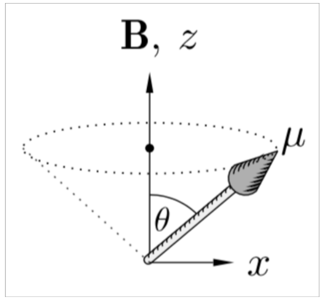
The Quantum Muon

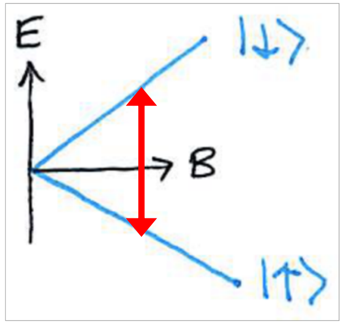


Lecture 2: Muonium

Stephen J. Blundell
University of Oxford

Quick recap:

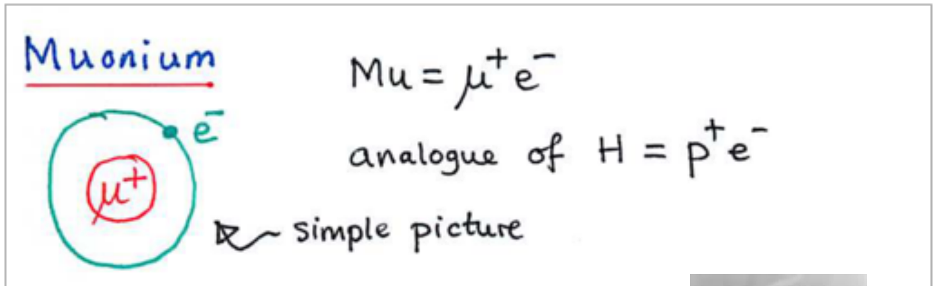


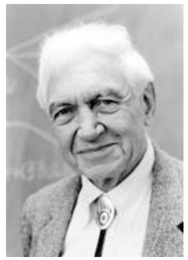


Muonium is a quantum object too!

The hyperfine coupling between the muon and the electron *entangles* their spin degrees of freedom.

First, what is muonium? What is a hyperfine interaction?
How to solve the Hamiltonian?
First, without the hyperfine interaction, then including it.



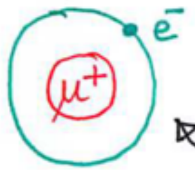


Vernon Hughes (1921-2003)

Muonium

$Mu = \mu^+ e^-$

analogue of $H = p^+ e^-$

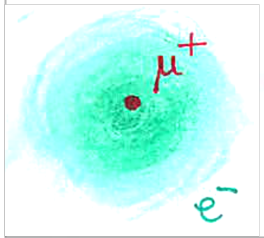


simple picture

electrically neutral

Mu^0

but two spins!



electron in 1s wavefunction

vacuum muonium

"isolated muonium atom in an empty universe"

Muonium is a two-spin system

⇒ four states to consider $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ ($\mu^+ e^-$)

$$\hat{H} = -\vec{\mu}_\mu \cdot \vec{B} - \vec{\mu}_e \cdot \vec{B} \qquad \vec{B} \parallel z$$

$$= \frac{\hbar\omega_e}{2}\sigma_e^z - \frac{\hbar\omega_\mu}{2}\sigma_\mu^z$$

$\omega_e = -\gamma_e B$ and $\omega_\mu = \gamma_\mu B$

$|\gamma_e|/2\pi = 28.024$ GHz T^{-1} $\qquad \gamma_\mu/2\pi = 135.5$ MHz T^{-1}

$$\omega_\pm = \frac{\omega_e \pm \omega_\mu}{2} \qquad \gamma_\pm = \frac{|\gamma_e| \pm \gamma_\mu}{2}$$

$$\frac{\gamma_+}{2\pi} = 14.08 \text{ GHz T}^{-1} \qquad \frac{\gamma_-}{2\pi} = 13.94 \text{ GHz T}^{-1}$$

$$m = \frac{\gamma_-}{\gamma_+} = 0.99037 \qquad \left(m \simeq 1 - \frac{2\gamma_\mu}{\gamma_e}\right)$$

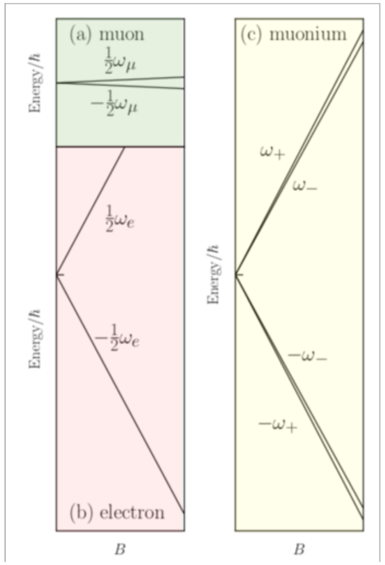
$$\{|\uparrow\mu\uparrow e\rangle, |\uparrow\mu\downarrow e\rangle, |\downarrow\mu\uparrow e\rangle, |\downarrow\mu\downarrow e\rangle\}$$

$$\sigma_\mu^z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma_e^z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\hat{H} = \frac{\hbar\omega_e}{2}\sigma_e^z - \frac{\hbar\omega_\mu}{2}\sigma_\mu^z$$

$$\omega_\pm = \frac{\omega_e \pm \omega_\mu}{2}$$

$$\hat{H}/\hbar = \begin{pmatrix} \omega_- & 0 & 0 & 0 \\ 0 & -\omega_+ & 0 & 0 \\ 0 & 0 & \omega_+ & 0 \\ 0 & 0 & 0 & -\omega_- \end{pmatrix}$$

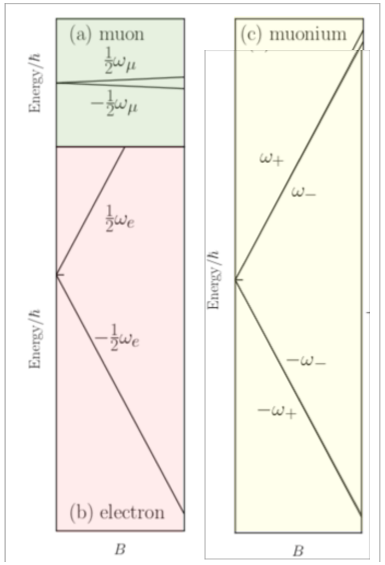


$$\omega_\pm = \frac{\omega_e \pm \omega_\mu}{2}$$

I have needed to exaggerate the difference between the muon and electron in this diagram (muon gyromagnetic ratio increased by factor of five)

$$\hat{H} = \frac{\hbar\omega_e}{2}\sigma_e^z - \frac{\hbar\omega_\mu}{2}\sigma_\mu^z$$

$$\hat{H}/\hbar = \begin{pmatrix} \omega_- & 0 & 0 & 0 \\ 0 & -\omega_+ & 0 & 0 \\ 0 & 0 & \omega_+ & 0 \\ 0 & 0 & 0 & -\omega_- \end{pmatrix}$$

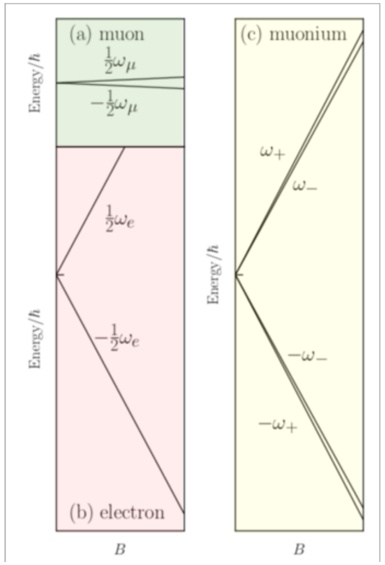


$$\omega_\pm = \frac{\omega_e \pm \omega_\mu}{2}$$

I have needed to exaggerate the difference between the muon and electron in this diagram (muon gyromagnetic ratio increased by factor of five)

$$\hat{H} = \frac{\hbar\omega_e}{2}\sigma_e^z - \frac{\hbar\omega_\mu}{2}\sigma_\mu^z$$

$$\hat{H}/\hbar = \begin{pmatrix} \omega_- & 0 & 0 & 0 \\ 0 & -\omega_+ & 0 & 0 \\ 0 & 0 & \omega_+ & 0 \\ 0 & 0 & 0 & -\omega_- \end{pmatrix}$$



$$\omega_\pm = \frac{\omega_e \pm \omega_\mu}{2}$$

I have needed to exaggerate the difference between the muon and electron in this diagram (muon gyromagnetic ratio increased by factor of five)

$$\hat{H} = \frac{\hbar\omega_e}{2}\sigma_e^z - \frac{\hbar\omega_\mu}{2}\sigma_\mu^z$$

$$\hat{H}/\hbar = \begin{pmatrix} \omega_- & 0 & 0 & 0 \\ 0 & -\omega_+ & 0 & 0 \\ 0 & 0 & \omega_+ & 0 \\ 0 & 0 & 0 & -\omega_- \end{pmatrix}$$

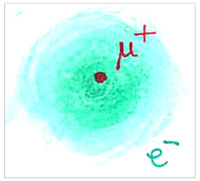
Local magnetic field at the muon site

* $B_L = \frac{\mu_0 M}{3}$ — LORENTZ FIELD
  site independent
  zero for antiferromagnets
* $B_{dip}(\underline{r}_\mu)$ — DIPOLAR FIELD
  depends on muon site
  depends on direction of $\underline{M}$
* $B_{hf}(\underline{r}_\mu)$ — HYPERFINE FIELD
  due to electron spin density at muon site
* B_{demag} — DEMAGNETIZATION FIELD
  depends on sample shape or domain structure

Hyperfine interaction in muonium

Hyperfine term

"A" $\hat{\vec{S}}_e \cdot \hat{\vec{S}}_\mu$ $\qquad \hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$



ISOTROPIC

Hyperfine interaction is *very weak*. As an energy, "*A*" is 18μeV. As a frequency, *A*=4.46 GHz.

$$\omega_0 = 2\pi A$$

$$\frac{\hat{H}}{\hbar} = \frac{1}{2}\omega_e\sigma_e^z - \frac{1}{2}\omega_\mu\sigma_\mu^z + \frac{1}{4}\omega_0 \boldsymbol{\sigma}_e \cdot \boldsymbol{\sigma}_\mu$$

ISOTROPIC

$$\{|{\uparrow_\mu}{\uparrow_e}\rangle, |{\uparrow_\mu}{\downarrow_e}\rangle, |{\downarrow_\mu}{\uparrow_e}\rangle, |{\downarrow_\mu}{\downarrow_e}\rangle\}$$

$$\frac{\hat{H}}{\hbar} = \frac{1}{4}\omega_0 \boldsymbol{\sigma}_e \cdot \boldsymbol{\sigma}_\mu$$

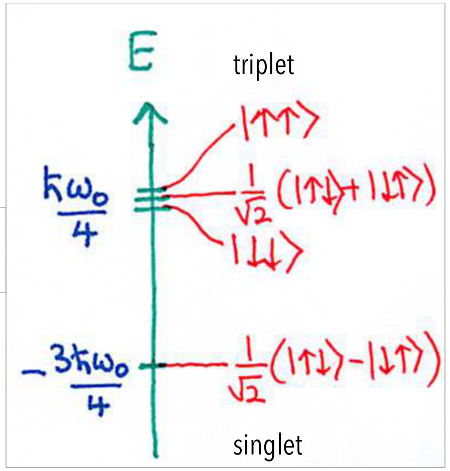
$$\boldsymbol{\sigma}_\mu \cdot \boldsymbol{\sigma}_e = \sigma_\mu^x\sigma_e^x + \sigma_\mu^y\sigma_e^y + \sigma_\mu^z\sigma_e^z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\frac{\hat{H}}{\hbar} = \begin{pmatrix} \frac{\omega_0}{4} & 0 & 0 & 0 \\ 0 & -\frac{\omega_0}{4} & \frac{\omega_0}{2} & 0 \\ 0 & \frac{\omega_0}{2} & -\frac{\omega_0}{4} & 0 \\ 0 & 0 & 0 & \frac{\omega_0}{4} \end{pmatrix}$$

$$\frac{\hat{H}}{\hbar} = \begin{pmatrix} \frac{\omega_0}{4} & 0 & 0 & 0 \\ 0 & -\frac{\omega_0}{4} & \frac{\omega_0}{2} & 0 \\ 0 & \frac{\omega_0}{2} & -\frac{\omega_0}{4} & 0 \\ 0 & 0 & 0 & \frac{\omega_0}{4} \end{pmatrix}$$

Eigenvalues

$$\frac{\omega_0}{4}, \frac{\omega_0}{4}, \frac{\omega_0}{4} \text{ and } -\frac{3\omega_0}{4}$$



- Two of the states are simple *product states* (they were in our original basis)
- The other two states are *entangled*

Now including the Zeeman terms the Hamiltonian becomes

$$\frac{\hat{H}}{\hbar} = \begin{pmatrix} \frac{\omega_0}{4} + \omega_- & 0 & 0 & 0 \\ 0 & -\frac{\omega_0}{4} - \omega_+ & \frac{\omega_0}{2} & 0 \\ 0 & \frac{\omega_0}{2} & -\frac{\omega_0}{4} + \omega_+ & 0 \\ 0 & 0 & 0 & \frac{\omega_0}{4} - \omega_- \end{pmatrix},$$

Eigenvalues

(1) $\frac{\omega_0}{4} + \omega_-$; (2) $\frac{\omega_0}{4} + \Omega$; (3) $\frac{\omega_0}{4} - \omega_-$; (4) $-\frac{3\omega_0}{4} - \Omega$

$$\Omega = \sqrt{(\tfrac{\omega_0}{2})^2 + \omega_+^2} - \frac{\omega_0}{2}$$

Eigenstates

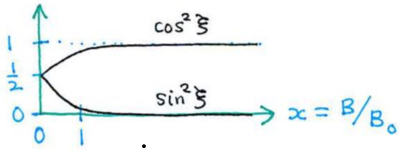
$|\uparrow\uparrow\rangle$ $\qquad \sin\xi|\uparrow\downarrow\rangle + \cos\xi|\downarrow\uparrow\rangle$ $\qquad |\downarrow\downarrow\rangle$ $\qquad \cos\xi|\uparrow\downarrow\rangle - \sin\xi|\downarrow\uparrow\rangle$

$$\sin\xi = \frac{1}{\sqrt{2}}\left(1 - \frac{x}{\sqrt{1+x^2}}\right)^{1/2}, \quad \cos\xi = \frac{1}{\sqrt{2}}\left(1 + \frac{x}{\sqrt{1+x^2}}\right)^{1/2}$$

$$x = 2\omega_+/\omega_0$$

$$B_0 = \omega_0/(|\gamma_e| + \gamma_\mu)$$

$$= 0.159\,\mathrm{T} \text{ for vacuum muonium}$$

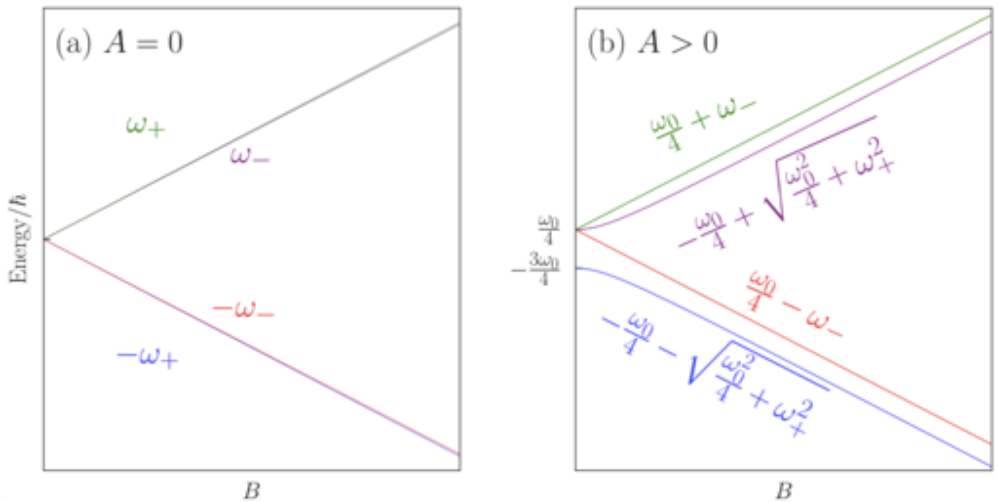


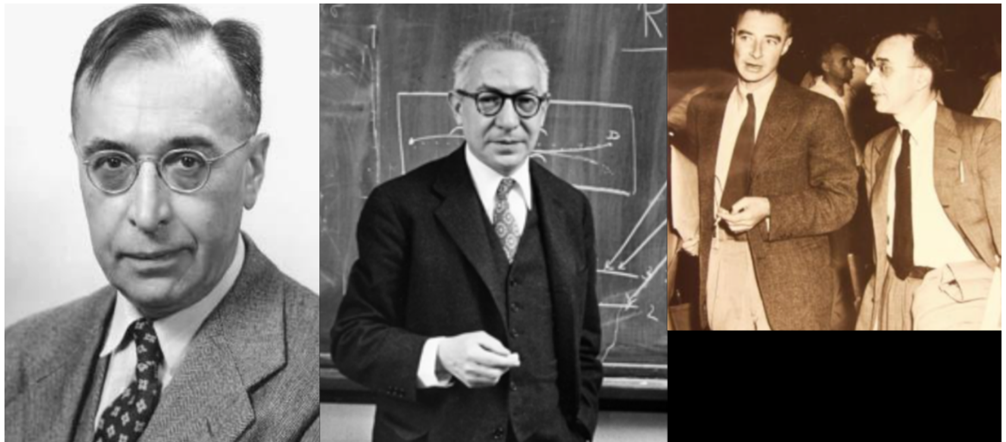
Eigenvalues

(1) $\frac{\omega_0}{4} + \omega_-$; (2) $\frac{\omega_0}{4} + \Omega$; (3) $\frac{\omega_0}{4} - \omega_-$; (4) $-\frac{3\omega_0}{4} - \Omega$.

Eigenstates

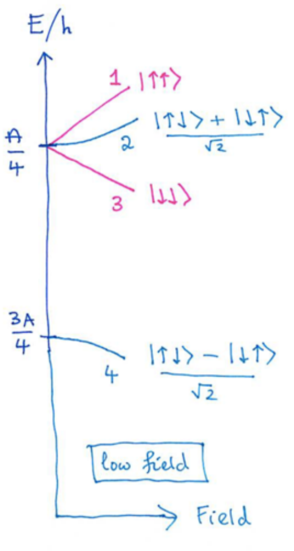
$|\uparrow\uparrow\rangle$ $\quad \sin\xi|\uparrow\downarrow\rangle + \cos\xi|\downarrow\uparrow\rangle$ $\quad |\downarrow\downarrow\rangle$ $\quad \cos\xi|\uparrow\downarrow\rangle - \sin\xi|\downarrow\uparrow\rangle$

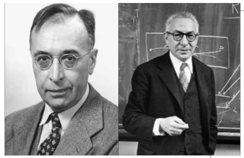


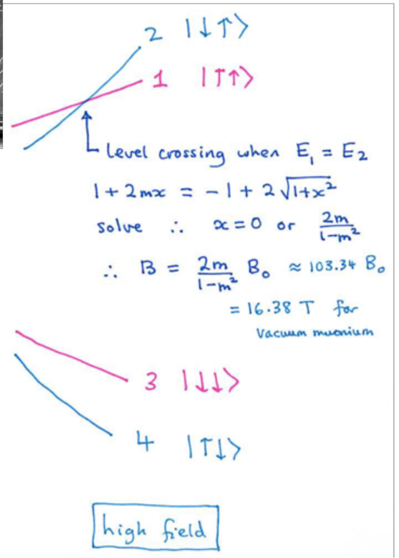


Gregory Breit
(1899-1981)

Isidor I. Rabi
(1898-1988)







- Muon polarized $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\left[\frac{1}{2}(1+\vec{\sigma}\cdot\hat{n})\right]$
- Electron unpolarized $\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ Combined density matrix $\rho = \frac{1}{4}(1+\vec{\sigma}\cdot\hat{n})$

$$= \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad \text{muon spin} \parallel z$$

$$P_\alpha(t) = \frac{1}{4} \mathrm{Tr}\left(\sigma_\mu^\alpha \, U(t) \, (1 + \sigma_\mu^z) \, U(t)^\dagger \right)$$

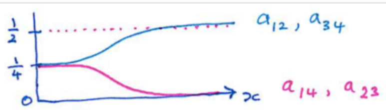
$\alpha = x, z$ (TF, LF)

$$= \frac{1}{4} \sum_{m,n} \langle m | \sigma_\mu^\alpha | n \rangle \langle n | \sigma_\mu^z | m \rangle e^{i\omega_{mn} t}$$

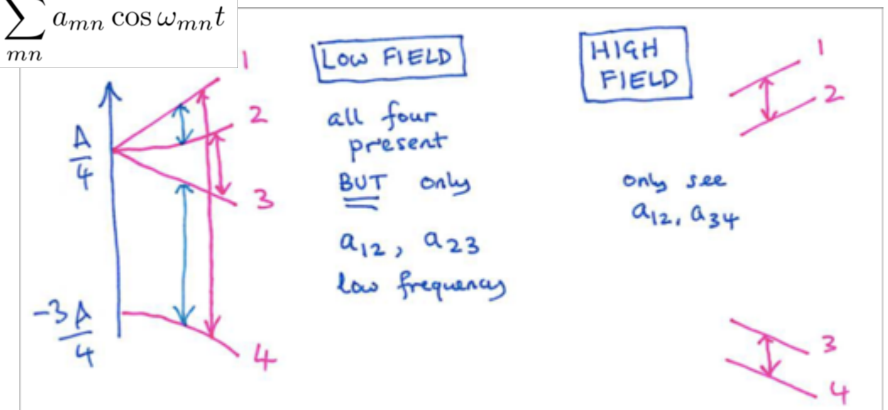
$$= \sum_{m,n} a_{mn} \cos \omega_{mn} t$$

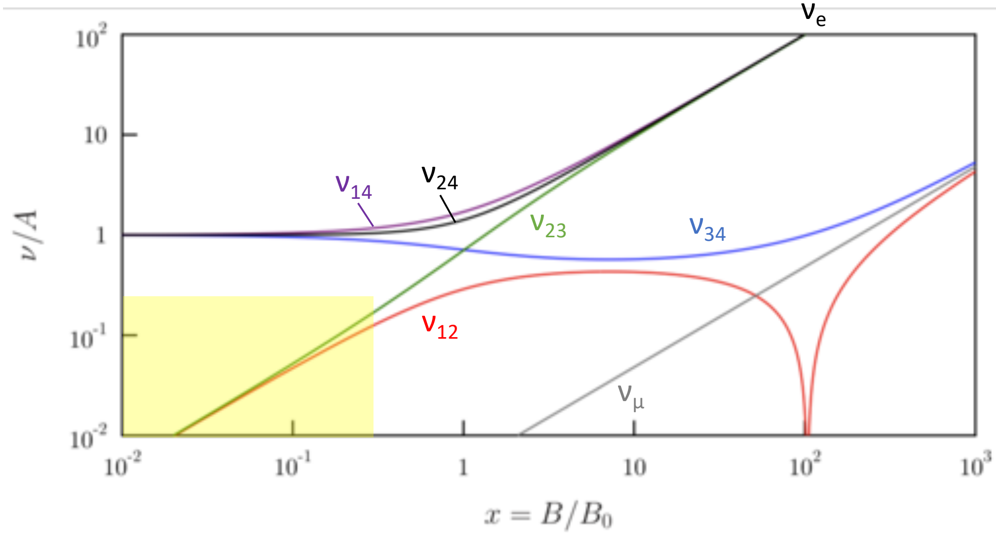
(with current Hamiltonian which has B//z)

a_{12}	a_{34}	a_{14}	a_{23}
$\frac{1}{2}\cos^2\xi$ $= \frac{1}{4}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$	$\frac{1}{2}\cos^2\xi$ $= \frac{1}{4}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$	$\frac{1}{2}\sin^2\xi$ $= \frac{1}{4}\left(1-\frac{x}{\sqrt{1+x^2}}\right)$	$\frac{1}{2}\sin^2\xi$ $= \frac{1}{4}\left(1-\frac{x}{\sqrt{1+x^2}}\right)$

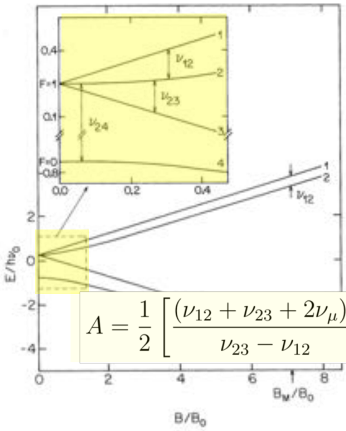


$$P(t) = \sum_{mn} a_{mn} \cos \omega_{mn} t$$





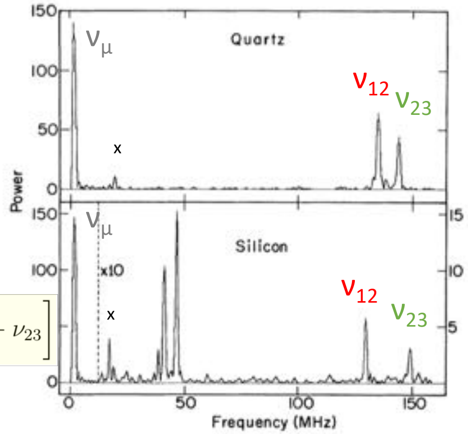
Low-field limit



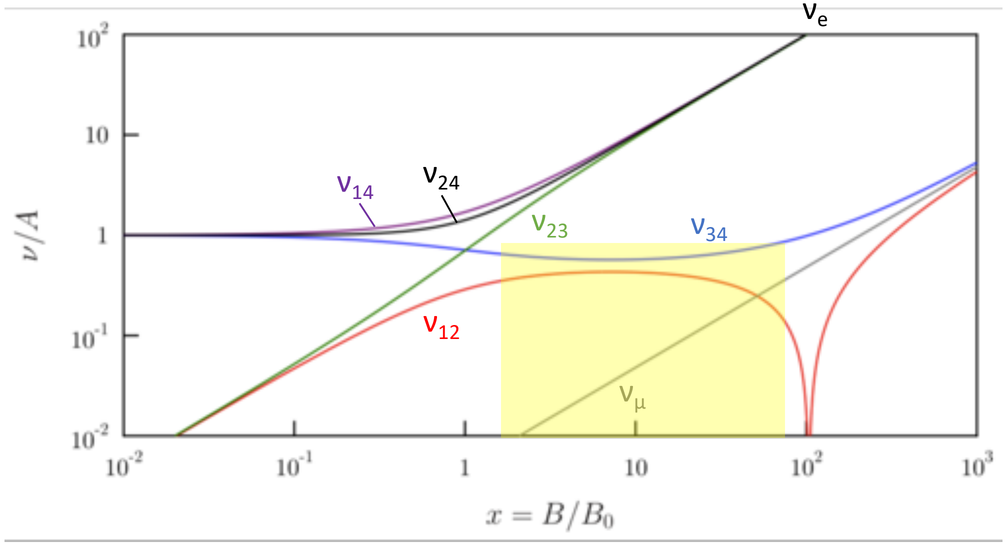
$$A = \frac{1}{2}\left[\frac{(\nu_{12}+\nu_{23}+2\nu_\mu)^2}{\nu_{23}-\nu_{12}} + \nu_{12} - \nu_{23}\right]$$

10 mT

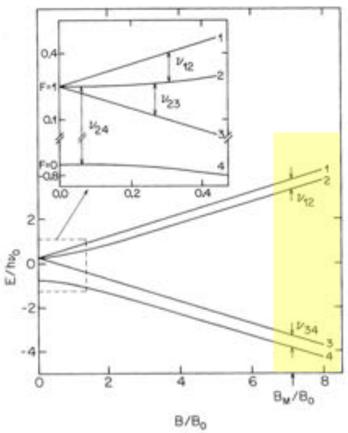
ν_μ=1.35 MHz, ν_e=280.2 MHz

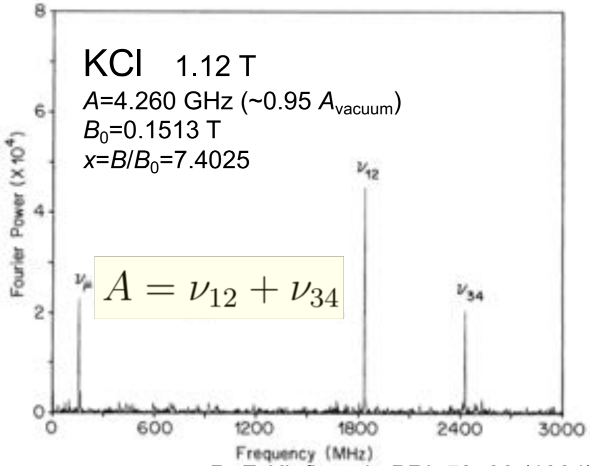


J. H. Brewer et al., PRL **31**, 143 (1973).



High-field limit

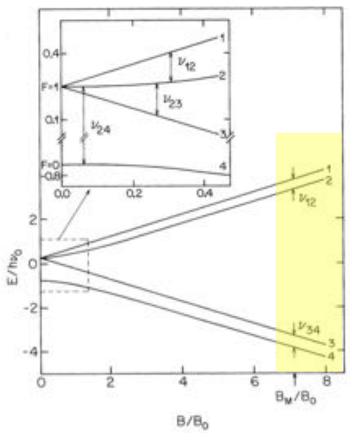




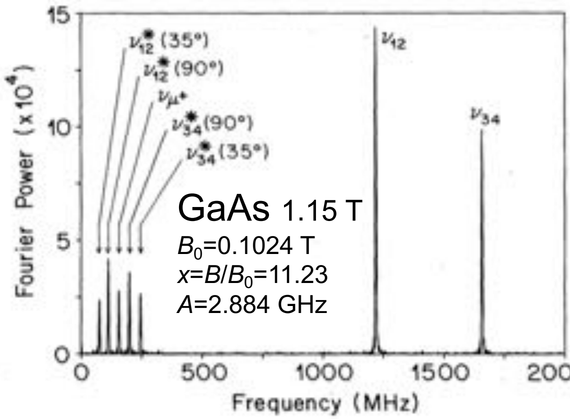
$$A = \nu_{12} + \nu_{34}$$

R. F. Kiefl et al., PRL **53**, 90 (1984).

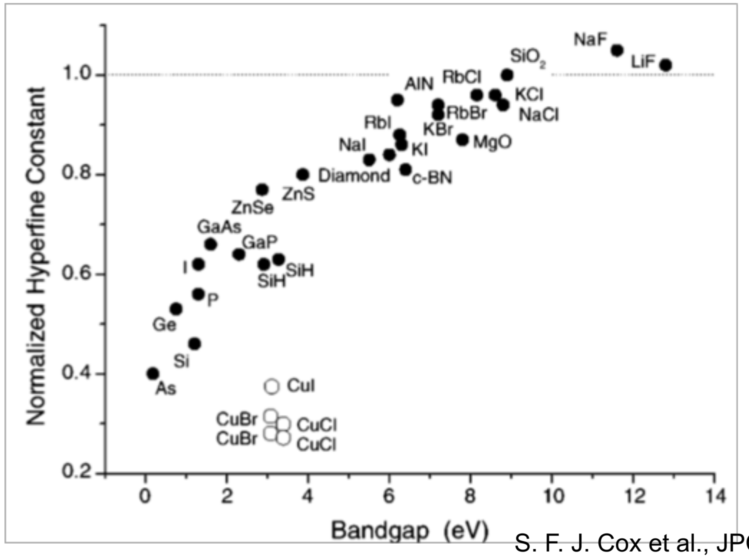
High-field limit



$$A = \nu_{12} + \nu_{34}$$



R. F. Kiefl et al., PRB **32**, 530 (1984).



S. F. J. Cox et al., JPCM **15**, R1727 (2003).